Math 4 Honors Name________________________________
Lesson 2-4: *Constructing Polynomial Function Models* Date ______________________________

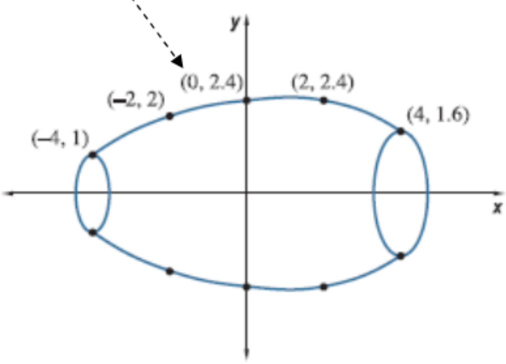
Learning Goal:

- *I can fit polynomial function models to data and graph patterns using problem conditions, statistical regression, and the method of undetermined coefficients (which is an application of matrices).*

When the first animated cartoons and movies were created, the apparent motion of every character was created by illustrators who made thousands of hand-drawn pictures, each slightly different than the one before.

Now most of that animation and many other visual effects in movies are created by computers. Creating computer images of the characters and action in such films often requires drawing and transforming smooth curves quickly.

The curves are specified by giving coordinates of key *control points*, as shown in this sketch of part of the arm from the robot in the image above.



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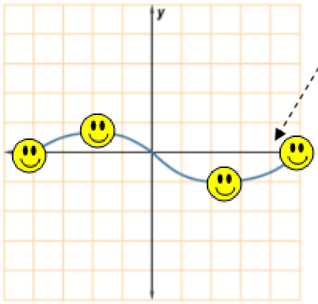
The partial model of a computer-animated figure was outlined by fairly simple curves, so you might try to model the shape with quadratic functions. You know from prior experience that coefficients for such a model can be determined by statistical curve-fitting software. You also know that finding functions whose graphs are more complex curves often requires polynomial expressions of degree three, four, or even higher. As you work on the problems of this investigation, look for answers to this question:

What efficient strategies will find polynomial functions with graphs that have two or more local maximum and minimum points?

The picture at the right shows a section of rural road with a kind of S-shaped curve leading back to the hills. To plan construction of such curvy roads, highway engineers find it helpful to make scale drawings of the planned road section and to represent sections of the road centerline by mathematical functions.

The following sketch shows such a planned road section on a coordinate grid. The coordinate axes have been placed on the road drawing in a way that helps with the modeling task.



Scale 1 unit = 100 feet

1. One way to begin the search for a function whose graph matches the S-shaped curve of the road is to focus attention on the points where the road curve crosses the x-axis.

   a. Why can you be sure that the function $f(x) = x(x + 4)(x - 5)$ has a graph with the same x-intercepts as the drawing of the S-shaped road curve?

   b. Produce a graph of $f(x)$ and compare it to the shape of the road curve. See if you can explain why the graph is such a poor model for the road shape, even though it has the same x-intercepts.

   c. What simple change in the rule for $f(x)$ would produce a better fit?

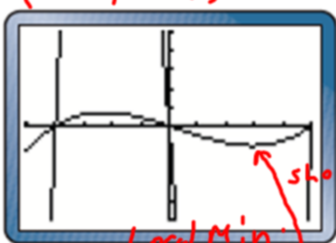
a. $f(x) = x(x+4)(x-5)$

$x = 0 \quad x = -4 \quad x = 5$

☆ Zero-Product Property

b. Local max & Local min are too high. We need coefficients for a better fit. Find the scale factor.

Local Max: $(-2.27, 28.55)$



should be here

Local Min: $(2.94, -42)$

C. Vertical shrink: $\approx \frac{-1}{-42} \approx .024$

$g(x) = .024(x)(x+4)(x-5)$

2. Choose a set of control points on the road curve graph that you believe will lead to a model for the shape that is better than that obtained in Problem 1. Then use those points and a cubic or quartic curve-fitting tool to find coefficients for a polynomial function whose graph is determined by those points.

Compare the graph of your new function model to the shape of the road curve and see if you can explain why the new graph is (or is not) a better match for the curve than the function in Problem 1.

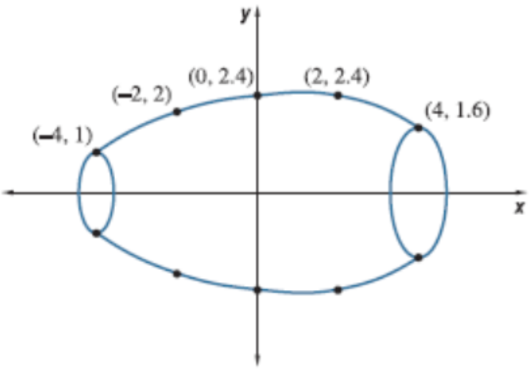
You will need to use at least 4 control points. Enter your points into a spreadsheet on your calculator. Run a regression . . .

(You may have different points.)

The points (-4, 0), (-2, .6), (2.5, -1) and (5, 0) give the equation

$y = .023x^3 + .0197x^2 + .4675x + .0708$

In some situations, like drawing the shape of a character in a computer-animated movie, close is not good enough. The graph of a function must pass directly through the prescribed control points. There is a third technique for constructing polynomial functions whose graphs pass through specific sets of control points. It is called the **method of undetermined coefficients**. To see how this method works, look again at the partial drawing of the arm from the robot shown in the introduction to this lesson.



It looks as if the curve on one side of the arm might be modeled well by the graph of a quadratic function and three control points would be enough to determine the coefficients in a rule like $g(x) = ax^2 + bx + c$.

3. Suppose that the control points chosen are (–4, 1), (0, 2.4), and (4, 1.6) and that they are used to determine the values of the coefficients a, b, and c of $g(x)$.

a. Explain why $g(-4) = 1$, $g(0) = 2.4$, and $g(4) = 1.6$.

Ordered pairs on the graph

b. Explain why the values of a, b, and c must satisfy all three of these equations:

$g(x) = ax^2 + bx + c$

$$16a - 4b + c = 1$$
$$0a + 0b + c = 2.4$$
$$16a + 4b + c = 1.6$$

when $x = -4$, $g(x) = 1$

$1 = a(-4)^2 + b(-4) + c$

$1 = 16a - 4b + c$

c. Adapt what you know about solving systems of linear equations with two unknowns to solve the system of three equations and three unknowns in Part b. ***Solve the system using substitution.***

$16a - 4b + c = 1$

$\cancel{0a} + \cancel{0b} + c = 2.4 \rightarrow c = 2.4$

$16a + 4b + c = 1.6$

$16a - \cancel{4b} + 2.4 = 1$

$+\ 16a + \cancel{4b} + 2.4 = 1.6$

$32a + 4.8 = 2.6$

$32a = -2.2$

$a = -.06875$

$16(-.06875) - 4b + 2.4 = 1$

$-1.1 - 4b + 2.4 = 1$

$1.3 - 4b = 1$

$-4b = -.3$

$b = .075$

d. What would the matrix equation look like to solve the system in 3b? Solve the system using the ***inverse-matrix method*** to verify what you found in part c.

$$\begin{bmatrix} 16 & -4 & 1 \\ 0 & 0 & 1 \\ 16 & 4 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 2.4 \\ 1.6 \end{bmatrix}$$

$$[A] \cdot X = [B]$$

$$X = [A]^{-1} [B]$$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -.06875 \\ .075 \\ 2.4 \end{bmatrix}$$

e. Write the rule for $g(x)$ implied by your results in Part c and compare its graph to the upper curve in the drawing.

$g(x) = -.06875x^2 + .075x + 2.4$

When you think about the fact that the modeling calculations in Problem 3 only deal with one part of a figure, you can easily imagine how animation work relies on powerful computers. The basic idea is illustrated fairly accurately by the example that you have worked on. With computers available, other mathematical strategies can be used to solve the resulting systems of linear equations.

4. Suppose that a highway engineer wanted to find a quartic function model for the S-shaped curve in Problems 1 and 2 and chose the control points (–4, 0), (–2, 0.5), (0, 0), (2, –1), and (5, 0). Answer the following questions to explain how those points could be used in the method of undetermined coefficients to find the desired quartic model.

a. The quartic function will be in the form $j(x) = ax^4 + bx^3 + cx^2 + dx + e$. What system of linear equations with unknowns a, b, c, d, and e would be implied by the requirement that the graph of $j(x)$ pass through the five chosen control points? One equation is given by

$$256a - 64b + 16c - 4d + e = 0.$$

$$\begin{cases} 256a - 64b + 16c - 4d + e = 0 \\ 16a - 8b + 4c - 2d + e = .5 \\ e = 0 \\ 16a + 8b + 4c + 2d + e = -1 \\ 625a + 125b + 25c + 5d + e = 0 \end{cases}$$

b. Translate the system from part a into a matrix equation. Write the matrix equation below.

Matrix Equation

$$\begin{bmatrix} 256 & -64 & 16 & -4 & 1 \\ 16 & -8 & 4 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 16 & 8 & 4 & 2 & 1 \\ 625 & 125 & 25 & 5 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = \begin{bmatrix} 0 \\ 0.5 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

c. Use the inverse-matrix method to solve the system. Write your solution matrix below.

$$\begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = \begin{bmatrix} .00248 \\ .02034 \\ -.07242 \\ -.45635 \\ 0 \end{bmatrix}$$

d. $j(x) = .00248x^4 + .02034x^3 - .07242x^2 - .45635x + 0$

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5. If you had to use the method of undetermined coefficients to find the equation of a 5th degree polynomial, what would be the minimum number of control points you would need? 6

- How many equations would make up the system you would have to solve? 6
- What key points might you choose for your control points?

x-intercepts, y-int., extrema

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Lesson 2-4 Homework

Show all work on another sheet of paper.

1. Use a system of equations to find the quadratic function that satisfies the quadratic function $f(x) = ax^2 + bx + c$ that satisfies the given equations. Use matrices to solve the system.

$$f(-2) = -15, f(-1) = 7, f(1) = -3$$

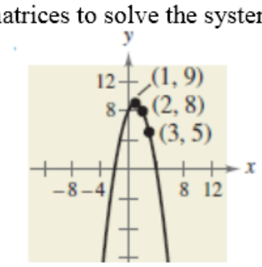
$f(x) =$ ______________________________

2. Use a system of equations to find ~~the quadratic~~ cubic function that satisfies the cubic function $f(x) = ax^3 + bx^2 + cx + d$ that satisfies the given equations. Use matrices to solve the system.

$$f(-2) = -7 \qquad f(1) = -4$$
$$f(-1) = 2 \qquad f(2) = -7$$

$f(x) =$ ______________________________

3. Use a system of equations to find the equation of the parabola $y = ax^2 + bx + c$ that passes through the points. Use matrices to solve the system. Graph the equation in your calculator to verify your results.



4. **DATA ANALYSIS: SNOWBOARDERS** The table shows the numbers of people y (in millions) in the United States who participated in snowboarding in selected years from 2003 to 2007. (Source: National Sporting Goods Association)

Year	Number, y
2003	6.3
2005	6.0
2007	5.1

(a) Use a system of equations to find the equation of the parabola $y = at^2 + bt + c$ that passes through the points. Let t represent the year, with $t = 3$ corresponding to 2003. Solve the system using matrices.

(b) Use a graphing utility to graph the parabola.

(c) Use the equation in part (a) to estimate the number of people who participated in snowboarding in 2009. Does your answer seem reasonable? Explain.

(d) Do you believe that the equation can be used for years far beyond 2007? Explain.

5. **ENROLLMENT** The table shows the enrollment projections (in millions) for public universities in the United States for the years 2010 through 2012. (Source: U.S. National Center for Education Statistics, *Digest of Education Statistics*)

Year	Enrollment projections
2010	13.89
2011	14.04
2012	14.20

(a) The data can be modeled by the quadratic function $y = at^2 + bt + c$. Create a system of linear equations for the data. Let t represent the year, with $t = 10$ corresponding to 2010.

(b) Use the matrix capabilities of a graphing utility to find the inverse matrix to solve the system from part (a) and find the least squares regression parabola $y = at^2 + bt + c$.

(c) Use the graphing utility to graph the parabola with the data.

(d) Do you believe the model is a reasonable predictor of future enrollments? Explain.

Lesson 2-4 HW Key

1. $f(x) = ax^2 + bx + c$

$4a - 2b + c = -15$

$a - b + c = 7$

$a + b + c = -3$

$$\begin{bmatrix} 4 & -2 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -15 \\ 7 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -9 \\ -5 \\ 11 \end{bmatrix}$$

$\boxed{f(x) = -9x^2 - 5x + 11}$

2. $f(x) = ax^3 + bx^2 + cx + d$

$-8a + 4b - 2c + d = -7$

$-a + b - c + d = 2$

$a + b + c + d = -4$

$8a + 4b + 2c + d = -7$

$$\begin{bmatrix} -8 & 4 & -2 & 1 \\ -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 \\ 8 & 4 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} -7 \\ 2 \\ -4 \\ -7 \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -4 \\ 1 \end{bmatrix}$$

$\boxed{f(x) = x^3 - 2x^2 - 4x + 1}$

3. $f(x) = ax^2 + bx + c$

$a + b + c = 9$

$4a + 2b + c = 8$

$9a + 3b + c = 5$

$$\begin{bmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \end{bmatrix}^{-1} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 8 \end{bmatrix}$$

$\boxed{f(x) = -x^2 + 2x + 8}$

4. $(3, 6.3)$ $(5, 6)$ $(7, 5.1)$

a. $9a + 3b + c = 6.3$
$25a + 5b + c = 6$
$49a + 7b + c = 5.1$

$$\begin{bmatrix} 9 & 3 & 1 \\ 25 & 5 & 1 \\ 49 & 7 & 1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 6.3 \\ 6 \\ 5.1 \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -.075 \\ .45 \\ 5.625 \end{bmatrix}$$

b. ✓

c. $S(9) = -.075(9)^2 + .45(9) + 5.625 = 3.6$ million people
$(2009 \Rightarrow t = 9)$

[Trend seems like #s are decreasing. Model yields larger #]

d. No. You are extrapolating. (Get negative values)

5. $t=10$ corresponds to 2010
$(10, 13.89)$ $(11, 14.04)$ $(12, 14.2)$

a. $100a + 10b + c = 13.89$
$121a + 11b + c = 14.04$
$144a + 12b + c = 14.2$

$$\begin{bmatrix} 100 & 10 & 1 \\ 121 & 11 & 1 \\ 144 & 12 & 1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 13.89 \\ 14.04 \\ 14.2 \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} .005 \\ .045 \\ 12.94 \end{bmatrix}$$

b. ✓

2013?

c. $e(13) = .005(13)^2 + .045(13) + 12.94$
$= 14.37$ million

Seems reasonable. Trend is increasing. S(13) yields a value that is more than 2012.